ON THE LENGTH OF PROGRAMS FOR COMPUTING FINITE BINARY SEQUENCES BY BOUNDED-TRANSFER TURING MACHINES

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Abstract 66T–26. G. J. Chaitin, The City College of the City University of New York, 819 Madison Avenue, New York, New York 10021. *On the length of programs for computing finite binary sequences by bounded-transfer Turing machines.* Preliminary report.

Consider Turing machines with one-way infinite tapes, n numbered internal states, the tape symbols blank, 0, and 1, starting in state 1 and halting in state n, and in which the number j of the next internal state after being in state i satisfies $|i-j| \leq b$. b can be chosen sufficiently large that any effectively computable infinite binary sequence is computable by such a machine. Such a Turing machine is said to compute a finite binary sequence S if starting with its tape blank and scanning the end square of the tape, it finally halts with S written at the end of the tape,

with the rest of the tape blank, and scanning the first blank square of the tape. Define $L(S)$ for any finite binary sequence S by: A Turing machine with n internal states can be programmed to compute S if and only if $n \geq L(S)$. Define $L(C_n)$ by $L(C_n) = \max L(S)$, where S is any binary sequence of length n. Let C_n be the set of all binary sequences of length n satisfying $L(S) = L(C_n)$.

Then

(1) $L(C_n) \sim an$.

(2) There exists a constant c such that for all m and n, those binary sequences S of length n satisfying

$$L(S) < L(C_n) - [\log_2 n] - m - c$$

are less than 2^{n-m} in number.

(3) For any $e > 0$ and $d > 1$, for all n sufficiently large, if S is a binary sequence of length n such that the ratio of the number of 0's in S to n differs from $\frac{1}{2}$ by more than e, then

$$L(S) < L(C_{[ndH(\frac{1}{2}+e,\frac{1}{2}-e)]}).$$

Here

$$H(p,q) = -p\log_2 p - q\log_2 q.$$

We propose also that elements of C_n be considered *the most patternless or random binary sequences of length n.* This leads to a definition and theory of randomness related to the R. von Mises–A. Wald–A. Church theory, but in accord with some criticisms of K. R. Popper.

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