

# Algorithmic Information Theory

## Some Recollections

Gregory Chaitin, 25 May 2007

### Introduction

AIT is a theory that uses the idea of the computer, particularly the size of computer programs, to study the limits of knowledge, in other words, what we can know, and how. This theory can be traced back to Leibniz in 1686, and it features a place in pure mathematics where there is absolutely no structure, none at all, namely the bits of the halting probability  $\Omega$ .

There are related bodies of work by other people going in other directions, but in my case the emphasis is on using the idea of algorithmic complexity to obtain incompleteness results. I became interested in this as a teenager and have worked on it ever since.

Let me tell you that story. History is extremely complicated, with many different points of view. What will make my account simple is the unity of purpose imposed on a field that is a personal creation, that has a central spine, that pulls a single thread. What did it feel like to do that? In fact, it's not something I did. It's as if the ideas wanted to be expressed through me.

It is an overwhelming experience to feel possessed by promising new ideas. This happened to me as a teenager, and I have spent the rest of my life trying to develop the ideas that flooded my mind then. These ideas were deep enough to merit 45 years of effort, and I feel that more work is still needed. There are many connections with crucial concepts in other fields: physics, biology, philosophy, theology, artificial intelligence... Let me try to remember what happened to me... The history of a person's life, that's just gossip. But the history of a person's ideas, that is real, that is important, that is where you can see creativity at work. That is where you can see new ideas springing into being.

## AIT in a Nutshell

Gödel discovered incompleteness in 1931 using a version of the liar paradox, “This statement is unprovable.” I was fascinated by Gödel’s work. I devoured Nagel and Newman, *Gödel’s Proof*, when it was published in 1958.

I was also fascinated by computers, and by the computer as a mathematical concept. In 1936 Turing derived incompleteness from uncomputability. My work follows in Turing’s footsteps, not Gödel’s, but adds the idea of looking at the size of computer programs.

For example, let’s call a program  $Q$  “elegant” if no program written in the same language that is smaller than  $Q$  produces the same output. Can we prove that individual programs are elegant? In general, no. Any given formal axiomatic system can only enable us to show that finitely many programs are elegant.

It’s easy to see that this must be so. Just consider a program  $P$  that calculates the output of the first provably elegant program that is larger than  $P$ .  $P$  runs through all the possible proofs in the formal axiomatic system until it finds the first proof that an individual program  $Q$  larger than  $P$  is elegant, and then  $P$  runs  $Q$  and returns  $Q$ ’s output as its ( $P$ ’s) output.

If you assume that only true theorems can be proved in your formal axiomatic system, then  $P$  is too small to be able to produce the same output as  $Q$ . If  $P$  actually succeeds in finding the program  $Q$ , then we have a contradiction. Therefore  $Q$  is never found, which means that no program that is bigger than  $P$  can be proven to be elegant.

So how big is  $P$ ? Well, it must include a big subroutine for running through all the possible proofs of the formal axiomatic system. The rest of  $P$ , the main program, is rather small;  $P$  is mostly that big subroutine. That’s the key thing, to focus on the number of bits in that subroutine.

So let’s define the algorithmic complexity of a formal axiomatic system to be the size in bits of the smallest program for running through all the proofs and producing all the theorems. Then we can state what we just proved like this: You can’t prove that a program is elegant if its size is substantially larger than the algorithmic complexity of the formal axiomatic system that you are using.

Instead of saying “a formal axiomatic system of algorithmic complexity  $N$ ,” I’ll just say “ $N$  bits of axioms.” So if you have  $N$  bits of axioms, then no program larger than  $N + c$  bits in size can be proven to be elegant. That’s the result we just proved.

A more sophisticated example is the number I call  $\Omega$ , which is the halting probability of a computer running a program produced one bit at a time by repeatedly tossing a coin. Because it is a probability, this number has to be between zero and one. Imagine writing it out in binary:

$$\Omega = .011100\dots$$

These bits are peculiar, they are irreducible mathematical information. This means that a formal axiomatic system with  $N$  bits of axioms can enable you to determine at most  $N + c$  bits of  $\Omega$ . Essentially the only way to determine bits of  $\Omega$  is to add that information directly to your axioms. Even though  $\Omega$  is a single well-defined real number (once you fix the programming language), its bits have no structure, no pattern, none at all, they are irredundant, irreducible mathematical information.

In other words, the bits of  $\Omega$  are mathematical facts that are true for no reason, no reason simpler than themselves.

So that's the basic idea, and those are my two favorite results, but the devil is in the details. You can spend your life on those details, and I did.

## Chaitin Research Timeline

- **1947:** Born in Chicago, child of Argentine immigrants. Family moves to New York.
- **1956:** Nagel and Newman's article on "Gödel's proof" is published in *Scientific American*. Article contains a photo by Arnold Newman of Gödel sitting in front of an empty blackboard at the Princeton Institute for Advanced Study.
- **1958:** Nagel and Newman's book *Gödel's Proof* is published by New York University Press.
- **1959:** Following directions in the *Scientific American* "Amateur Scientist" department, I build a Van de Graaff generator for high-voltage static electricity.
- **1962:** First year at Bronx High School of Science. While answering an essay question on the entrance exam for the Columbia University

Science Honors Program for bright high school students, I get the idea of defining randomness using program-size complexity.

The essay question is what do you conclude if you find a pin on the moon.<sup>1</sup> My answer is that this means that somebody must have visited before you, because a pin is not natural, it is artificial, the product of intelligence. And, I remark, what this means is that there is a small program to calculate it, to create it. That's how we can tell that the pin has structure and is artificial. And, contrariwise, something natural would not have a description that can be compressed into a small program, because it was not designed.

And then, as a throw-away remark, I state that a random thing is one that cannot be compressed into a smaller program. More precisely, I am speaking about a digital description of an object, not about the object itself. In other words, in 1962 I give the following

- **Definition of Randomness R1:** A random finite binary string is one that cannot be compressed into a program smaller than itself, that is, that is not the unique output of a program without any input, a program whose size in bits is smaller than the size in bits of its output.

However I quickly forget about this definition, because I am having so much fun learning how to write, debug and run computer programs in the Science Honors Program. And I am given the run of the math stacks at Columbia University and can hold in my hands and study the collected works of Euler and other priceless volumes.

- **1963:** Shannon and McCarthy, *Automata Studies*, Princeton University Press, 1956, contains E. F. Moore's paper "Gedanken-experiments on sequential machines."<sup>2</sup> Following Moore, I write a program for identifying a finite-state black box by putting in inputs and looking at the outputs. My experiments suggest this is easier to do than Moore anticipated. I prove this to be the case in a note "An improvement on a theorem of E. F. Moore" (*IEEE Transactions on Electronic Computers*, 1965), my first publication.

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<sup>1</sup>This was before the first lunar landing.

<sup>2</sup>I became aware of Shannon and McCarthy, perhaps the first book on the theory of computation, because it was reviewed in *Scientific American*.

- **1964:** Summer vacation between high school and college, I try to find an infinite set with no subset that is easier to generate than the entire set. By easier I mean faster or simpler; at this point I am simultaneously exploring run-time complexity and program-size complexity. The work goes well, but is not published until 1969 in the *ACM Journal* as “On the simplicity and speed of programs for computing infinite sets of natural numbers.”

Also that summer, I get the first incompleteness result that I will publish, **UB1**, an upper bound on the provable lower bounds on run-time complexity in any given formal axiomatic system. This is published in 1970 in a Rio de Janeiro Pontifícia Universidade Católica research report, and only there.<sup>3</sup>

Another discovery that summer, **UB2**, is that one can diagonalize over the output of all programs that provably calculate total functions  $f: N \rightarrow N$  to obtain a faster growing computable total function  $F: N \rightarrow N$ . That is to say, given any formal axiomatic system, one can construct a computer program from it that calculates a total function  $f: N \rightarrow N$ , but the fact that this program calculates a total function  $f: N \rightarrow N$  cannot be proved within the formal axiomatic system, because  $f$  goes to infinity too quickly. “Calculates a total function  $f: N \rightarrow N$ ” merely means that every time we give the program  $f$  a natural number  $n$  as input, it eventually outputs a single natural number  $f(n)$  and then halts.

The result UB1 is actually a corollary of UB2, since all lower bounds on run-time complexity are computable total functions.

Now one would say that the proof of UB2 is an instance of Cantor diagonalization, but in my opinion it’s really closer to Paul du Bois-Reymond’s theorem on orders of infinity. His theorem is that for any scale of rates of growth, any infinite list of functions that go to infinity faster and faster, for example

$$\begin{aligned} f_0(n) &= 2^n, \\ f_1(n) &= 2^{2^n}, \\ f_2(n) &= 2^{2^{2^n}} \dots, \end{aligned}$$

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<sup>3</sup>While writing up that report in Rio, I realize I can also obtain an upper bound on the provable lower bounds on program-size complexity.

there is another function

$$f_\omega(n) = \max_{k \leq n} f_k(n)$$

that goes to infinity even more quickly. As far as I know, Paul du Bois-Reymond's work was independent of Cantor's.

Note the Cantor ordinal number  $\omega$  as a subscript. We can then form

$$\begin{aligned} f_{\omega+1}(n) &= 2^{f_\omega(n)}, \\ f_{\omega+2}(n) &= 2^{f_{\omega+1}(n)}, \\ f_{\omega+3}(n) &= 2^{f_{\omega+2}(n)} \dots \end{aligned}$$

and then

$$f_{2\omega}(n) = \max_{k \leq n} f_{\omega+k}(n).$$

Continuing in this manner, we get to

$$f_{3\omega} \dots f_{4\omega} \dots f_{\omega^2} \dots f_{\omega^3} \dots f_{\omega^\omega} \dots f_{\omega^{\omega^\omega}} \dots$$

and onwards and upwards, incredibly fast-growing functions all.

I had read about Paul du Bois-Reymond's work in a monograph by G. H. Hardy called *Orders of Infinity*.<sup>4</sup>

My point of view changes between 1964 and 1965. In my 1964 work, only **infinite** computations are considered. In 1965, on the contrary, only **finite** computations are considered, computations that produce a **single output**. Furthermore, my interest shifts from run-time complexity to program-size complexity.

- **1965:** During the spring term of my first year at City College, CUNY, I simultaneously study three books: von Neumann and Morgenstern, *Theory of Games and Economic Behavior*, Shannon and Weaver, *The Mathematical Theory of Communication*, and Turing's 1936 paper "On computable numbers. . ." in the anthology Davis, *The Undecidable*. The 1962 definition of randomness (R1) comes back to me; as I will now explain, all three books play a vital role. It all comes together as I am

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<sup>4</sup>I learned the calculus from Hardy's *A Course of Pure Mathematics*, and also enjoyed *A Mathematician's Apology* and Hardy and Wright, *An Introduction to the Theory of Numbers*.

reading a discussion in von Neumann and Morgenstern of the game of matching pennies, for which their theory says that you should toss a coin.<sup>5</sup>

In a footnote they remark that the theory of games seems to require a quantum-mechanical world in which God plays dice. Not really, I say to myself. Another logical possibility would be that the theory of games tells you to use a random sequence of choices, but you cannot compute this sequence of choices from the theory.

You see, in the game of matching pennies, if a theory can tell you exactly what to do, you can predict what your opponent will do and beat him. The solution to the paradox is either that the theory asks you to use physical randomness, which is unpredictable, or that you have a theory that says you must make mathematically random or unstructured choices, and the contradiction is avoided because these are in fact uncomputable (a notion taken from Turing).

The third leg of the stool comes from reading Shannon, who defines a message to be random or have maximum entropy if it cannot be compressed, if it cannot be encoded more compactly. Obviously the most general possible **decoder** would be a universal Turing machine, a general-purpose computer.

With my 1962 definition (R1), I have managed to connect game theory, information theory and computability theory. Now all I have to do is work out the details.

Now it's the summer vacation between my first and second year at City College, and I attempt to carry out the plan. At the beginning of the summer, the road forward seems blocked, but I keep trying. Later in the summer, ideas begin to flood into my mind. I write a single paper that is the size of a small book. At the request of the editors, I later divide it in two, and delete much material to save space.

This paper “On the length of programs for computing finite binary sequences”—part one is published in 1966 and part two is published in 1969, both in the *ACM Journal*—presents **three** different theories of program-size complexity (and embryonic versions of ideas that I would explore for years):

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<sup>5</sup>One of the players is trying to match the other player's choice of head or tails.

- **Complexity Theory (A)**: Counting the number of states in a normal Turing machine with a fixed number of tape symbols. I call this Turing machine state-complexity.
- **Complexity Theory (B)**: The same as theory (A), but now there’s a fixed upper bound on the size of transfers—jumps, branches—between states. You can only jump nearby. I call this bounded-transfer Turing machine state-complexity.
- **Complexity Theory (C)**: Counting the number of bits in a binary program, a bit string. The program starts with a **self-delimiting** prefix, indicating which computing machine to simulate, followed by the binary program for that machine. That’s how we get what’s called a **universal machine**.<sup>6</sup>

Let’s define the complexity of a bit string to be the size of the smallest program that computes it.

In each case, theory (A), (B) or (C), I show that most  $n$ -bit strings have complexity close to the maximum possible, and I determine asymptotic formulas for the maximum possible complexity of an  $n$ -bit string. These maximum or near maximum complexity strings are defined to be random. To show that this is reasonable, I prove, for example, that these strings are “normal” in Borel’s sense. This means that all possible blocks of bits of the same size occur in such strings approximately the same number of times, an equi-distribution property.

I start with theory (A) because that seems the most straightforward thing to do. The idea in theory (A) is to eliminate all the redundancy in a real programming language. Then I switch to theory (B), in which I don’t eliminate the redundancy, I live with it. The proofs are prettier; more subtle, not so heavy-handed. However, in theories (A) and (B) I cannot figure out how to show that a **small** amount of structure in an  $n$ -bit string will force its complexity to dip below the maximum possible complexity for  $n$ -bit strings.

To solve this, I switch from Turing machines to binary programs and theory (C). Theory (C) solves the problem, but feels too easy, like stealing candy from a baby. For example, in theory (C) it is trivial

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<sup>6</sup>I call theory (C) “blank-endmarker” program-size complexity, to distinguish it from “self-delimiting” program-size complexity, theory (D) below.

to show that most  $n$ -bit strings have close to the maximum possible complexity. And this maximum possible complexity is precisely  $n + 1$ , not an asymptotic estimate as in theories (A) and (B).<sup>7</sup>

By the way, in theories (A) and (B), randomness definition (R1) does not apply, because the size of programs is measured in states, not bits. It is necessary to use a slightly different definition of randomness:

- **Definition of Randomness R2:** A random  $n$ -bit string is one that has maximum or near maximum complexity. In other words, an  $n$ -bit string is random if its complexity is approximately equal to the maximum complexity of any  $n$ -bit string.

In theory (C), (R1) works fine (but so does (R2), which is more general). Theory (C) is essentially the same as the one independently proposed by Kolmogorov at about the same time (1965).<sup>8</sup>

However, I am dissatisfied with theory (C); the absence of **subadditivity** disturbs me. What is subadditivity? The usual definition is that a function  $f$  is subadditive if  $f(x + y) \leq f(x) + f(y)$ . I mean something slightly different. Subadditivity holds if the complexity of computing two objects together (also known as their **joint complexity**) is bounded by the sum of their individual complexities.<sup>9</sup> In other words, subadditivity means that you can combine subroutines by concatenating them, without having to add information to indicate where the first subroutine ends and the second one begins. This makes it easy to construct big programs. Complexity is subadditive in theories (A) and (B), but not in theory (C).

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<sup>7</sup>To get (max complexity  $n$ -bit string) =  $n + 1$ , theory (C) has to be a bit more complicated.

- **Complexity Theory (C2):** If the first bit of the program is a 0, then output the rest of the program as is and halt. If the first bit of the program is a 1, process the rest of the program as in theory (C).

(C2) is the version of theory (C) given in “On the length of programs for computing finite binary sequences: statistical considerations” (*ACM Journal*, 1969), the second of the two papers put together from my 1965 randomness manuscript.

<sup>8</sup>Solomonoff was the first person to publish the idea of program-size complexity—in fact, (C)—but he did not propose a definition of randomness.

<sup>9</sup>In the case of joint complexity the computer has **two** outputs, or outputs **a pair** of objects, whatever you prefer.

Last but not least, “On the length of programs for computing finite binary sequences” contains what I would now call a Berry paradox proof that program-size complexity is uncomputable. This seed was to grow into my 1970 work on incompleteness, where I refer to the Berry paradox explicitly for the first time.

- **1966:** Awarded by City College the Belden Mathematical Prize and the Gitelson Medal “for the pursuit of truth.” Family moves back to Buenos Aires.
- **1967:** I join IBM Argentina, working as a computer programmer.
- **1969:** Stimulated by von Neumann’s posthumous *Theory of Self-Reproducing Automata*, I work on a mathematical definition of life using program-size complexity. This is published in Spanish in Buenos Aires, and the next year (1970) in English in the *ACM SICTACT News*. This is the first of what on the whole I regard as an unsuccessful series of papers on theoretical biology.<sup>10</sup>
- **1970:** I visit Brazil and inspired by this tropical land, I realize that one can get powerful incompleteness results using program-size arguments. In fact, one can place **upper** bounds on the provable **lower** bounds on run-time **and** program-size complexity in a formal axiomatic system. And this provides a way to measure the power of that formal axiomatic system.

This first information-theoretic incompleteness result is immediately published in a Rio de Janeiro Pontificia Universidade Católica research report and also as an *AMS Notices* abstract, and comes out the next year (1971) as a note in the *ACM SIGACT News*.

I obtain a *LISP 1.5 Programmer’s Manual* in Brazil and start writing LISP interpreters and inventing LISP dialects.<sup>11</sup>

- **1971:** I write a longer paper on incompleteness, “Information-theoretic limitations of formal systems,” which is presented at the Courant Institute Computational Complexity Symposium in New York City in

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<sup>10</sup>The latest one is “Speculations on biology, information and complexity” (*EATCS Bulletin*, February 2007).

<sup>11</sup>Around 1973, I give courses on LISP and on computability and metamathematics at the University of Buenos Aires.

October 1971. A key idea in this paper is to measure the complexity of a formal axiomatic system by the size in bits of the program that generates all of the theorems by systematically running through the tree of all possible proofs.

- **1973:** I complete a greatly expanded version of “Information-theoretic limitations of formal systems.” The expanded version appears in the *ACM Journal* in 1974. A less technical paper on the same subject, “Information-theoretic computational complexity,” is presented at the IEEE International Symposium on Information Theory, in Ashkelon, Israel, June 1973, and is published in 1974 as an invited paper in the *IEEE Transactions on Information Theory*.<sup>12</sup>
- **1974:** I am invited to visit the IBM Watson Lab in Yorktown Heights for a few months. The visit goes well, with a number of major breakthroughs. I realize what to do to theory (C) to restore subadditivity, and discover the halting probability  $\Omega$ .
  - **Complexity Theory (D):** Counting the number of bits in a self-delimiting binary program, a bit string with the property that you can tell where it ends by reading it bit by bit without ever reading a blank endmarker. Now a program starts with a self-delimiting prefix as before, but the program to be simulated that follows the prefix must **also** be self-delimiting. So the idea is that the **whole** program must now have the same property the **prefix** already had in theory (C).

(D) is the mature theory, I believe. I immediately put this out as an IBM research report. I present this paper at the opening plenary session of the IEEE International Symposium on Information Theory in Notre Dame, Indiana, in October 1974. It is published in the *ACM Journal* in 1975 as “A theory of program size formally identical to information theory.”

There are three key ideas in this paper: self-delimiting programs, a new definition of relative complexity, and the idea of getting program-size results **indirectly** from probabilistic, measure-theoretic arguments

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<sup>12</sup>In 1974 I send a copy of this paper to Kurt Gödel, leading to a pleasant, short phone conversation with Gödel and an appointment to meet him at the Princeton Institute for Advanced Study, an appointment that Gödel cancels at the last minute.

involving the probability  $P(x)$  that a program will calculate  $x$ . I call this the algorithmic probability of  $x$ .<sup>13</sup> Summing  $P(x)$  over all possible outputs  $x$  yields the halting probability  $\Omega$ :

$$\Omega = \sum_x P(x).^{14}$$

And a key theorem

$$(*) \quad H(x) = -\log_2 P(x) + O(1)$$

permits us to translate complexities into probabilities and vice versa. Here the complexity  $H(x)$  of  $x$  is the size in bits of the smallest program for calculating  $x$ , and the  $O(1)$  indicates that the difference between the two sides of the equation is bounded.

Incidentally,  $(*)$  implies that there are few minimum or near-minimum size programs for calculating something, few minimal descriptions. That is, an elegant program for calculating something is essentially unique.<sup>15</sup>

Where did the halting probability  $\Omega$  come from? How did I come up with it? Well, already in part two of my first paper on randomness, “On the length of programs for computing finite binary sequences: statistical considerations” (*ACM Journal*, 1969), I use a Heine-Borel-style algorithm to exhibit a specific example of a random infinite sequence of bits. I think it is important to come up with specific examples.

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<sup>13</sup>Solomonoff tried to define  $P(x)$  but could not get  $P(x)$  to converge since he wasn’t working with self-delimiting programs.

<sup>14</sup>This definition is a bit abstract. Here are two other ways of defining an  $\Omega$  number. As a sum over programs  $p$ :

$$\Omega' = \sum_{p \text{ halts}} 2^{-|p|}.$$

As a sum over all positive integers  $n$ :

$$\Omega'' = \sum_n 2^{-H(n)}.$$

Here  $|p|$  is the size in bits of the program  $p$ , and  $H(n)$  is the size in bits of the smallest program for calculating the positive integer  $n$ .

<sup>15</sup>For more on this, see my book *Exploring Randomness* (2001). I had proven this result in theory (C) in 1972, but that proof wasn’t published until 1976, in “Information-theoretic characterizations of recursive infinite strings” in *Theoretical Computer Science*.

The halting probability  $\Omega$  is a natural example of a random infinite sequence of bits. Besides providing a connection with the work of Turing,  $\Omega$  makes randomness more concrete and more believable.

Furthermore, once you have a natural example of randomness, you immediately get an incompleteness theorem from it, as I point out in “Gödel’s theorem and information” (*International Journal of Theoretical Physics*, 1982). My work in 1987 on  $\Omega$  and its diophantine equation makes this fully explicit. I had been aware of this opportunity for getting a dramatic incompleteness result for some time.<sup>16</sup>

(\*) is nice but it is only part of the story. The true reward for changing from theory (C) to (D) is this spectacular result:

$$(**) \quad H(x, y) = H(x) + H(y|x) + O(1).$$

In words, (the joint complexity of two objects) is equal to the sum of (the absolute complexity of the first object) plus (the relative complexity of the second object given the first object).

To get (\*\*), self-delimiting programs aren’t enough, you also need the right definition of **relative complexity**.<sup>17</sup> I had used relative complexity in my big 1965 randomness manuscript, but had eliminated it to save space. In my 1975 *ACM Journal* paper I take up relative complexity again, but define  $H(x|y)$ , the complexity of  $x$  given  $y$ , to be the size in bits of the smallest self-delimiting program for calculating  $x$  if we are given for free, not  $y$  directly, but a minimum-size self-delimiting program for  $y$ .

And (\*\*) implies that the extent to which computing two things together is cheaper than computing them separately, also known as the mutual information

$$H(x : y) \equiv H(x) + H(y) - H(x, y),$$

is essentially the same, within  $O(1)$ , of the extent to which knowing  $x$  helps us to know  $y$

$$H(y) - H(y|x),$$

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<sup>16</sup>See the end of the introductory section of “Information-theoretic limitations of formal systems” (*ACM Journal*, 1974).

<sup>17</sup>Levin claims to have published theory (D) first. However he missed this vital part of theory (D).

and this in turn is essentially the same, within  $O(1)$ , of the extent to which knowing  $y$  helps us to know  $x$

$$H(x) - H(x|y).$$

This is so pretty that I decide never to use theory (C) again. For (D) doesn't just restore subadditivity to (C), it reveals an elegant new landscape with sharp results instead of messy error terms. From now on, theory (D) only.

- **1975:** My first *Scientific American* article, “Randomness and mathematical proof,” appears. I move back to New York and join the IBM Watson Lab.

In the paper “Algorithmic entropy of sets” (*Computers & Mathematics with Applications*, 1976, written at the end of 1975), I attempt to extend the self-delimiting approach to programs for generating infinite sets of output. Much remains to be done.<sup>18</sup>

This topic is important, because I think of a formal axiomatic system as a computation that produces theorems. My measure of the complexity of a formal axiomatic system is therefore the size in bits of the smallest **self-delimiting** program for generating the infinite set of theorems.

- **1976–1985:** I concentrate on software and hardware engineering for IBM's RISC (Reduced Instruction Set Computer) project.

Even though I spend most of my time on this engineering project, in 1982 I publish “Gödel's theorem and information” in the *International Journal of Theoretical Physics*.<sup>19</sup> This is later included with my 1974 *IEEE Transactions on Information Theory* paper in the influential anthology Tymoczko, *New Directions in the Philosophy of Mathematics*, Princeton University Press, 1998.

My 1985 publication “An APL2 gallery of mathematical physics: a course outline” gives computational working models of physical phenomena to be inserted between chapters of Einstein and Infeld, *The*

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<sup>18</sup>If this interests you, please see the discussion of infinite computations in the last chapter of *Exploring Randomness* (2001).

<sup>19</sup>At roughly the same time I fulfill a childhood dream by building my own telescope: I join a telescope-making club and grind a 6 inch f/8 mirror for a Newtonian reflector in a basement workshop at the Hayden Planetarium of the Museum of Natural History.

*Evolution of Physics.* This APL2 physics simulation software is extremely concise.<sup>20</sup>

- **1986:** My RISC engineering work stops because of an invitation by Cambridge University Press to write the first volume in their series Cambridge Tracts in Theoretical Computer Science. I start working on the book, intending merely to collect previous results, but then the flow of new ideas resumes.

Some of these new ideas are presented in the paper “Incompleteness theorems for random reals” (1987). This paper contains a proof of the basic result that an  $N$ -bit formal axiomatic system cannot enable you to determine more than  $N + c$  bits of  $\Omega$ , bits that may be scattered and do not have to be together at the beginning of  $\Omega$ . This is the measure-theoretic proof that I give in the Cambridge book. After writing this paper, work on the book begins in earnest.

Using work by Jones and Matijasevic on Hilbert’s 10th problem, I calculate a 200-page diophantine equation for  $\Omega$ .<sup>21</sup> This equation has thousands of unknowns and a parameter  $k$ , and has finitely or infinitely many whole-number solutions depending on whether the  $k$ th bit of  $\Omega$  is, respectively, a 0 or a 1. To get this equation, I convert a register machine program for a LISP interpreter into a diophantine equation, and then I plug into that equation a LISP program for computing lower bounds on  $\Omega$ .

- **1987:** Cambridge University Press publishes *Algorithmic Information Theory*, which explains how to obtain the diophantine equation for  $\Omega$ . This book also contains a result about random infinite binary sequences. I show that four definitions of this concept are equivalent: two constructive measure-theoretic definitions due to Martin-Löf and to Solovay, and two definitions of my own using program-size complexity. *Algorithmic Information Theory* is my first publication in which LISP appears.

Simultaneously, World Scientific publishes a collection of my papers, *Information, Randomness and Incompleteness*.

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<sup>20</sup>Besides learning the physics and some numerical analysis, I wanted to get a feel for the algorithmic complexity of the laws of physics.

<sup>21</sup>It’s actually what’s called an **exponential** diophantine equation.

- **1988:** I write about the diophantine equation for  $\Omega$  in my second *Scientific American* article, “Randomness in arithmetic” (1988), and then in *New Scientist* (1990), and after that in *La Recherche* (1991).
- **1991:** I give a lecture on “Randomness in arithmetic” in the room where Gödel taught at the University of Vienna.
- **1992:** In the 1992 paper on “Information-theoretic incompleteness,” I publish a program-size proof of the theorem that you cannot determine the bits of  $\Omega$ . More precisely, an  $N$ -bit theory—a formal axiomatic system with complexity  $N$ —can permit you to determine at most  $N + c$  bits of  $\Omega$ . This program-size proof is better than the measure-theoretic proof in the 1987 Cambridge book, and is the proof that I use in the 1998 book, *The Limits of Mathematics*.

I also publish four papers on LISP program-size complexity:

- **Complexity Theory (L):** Counting the number of characters a LISP S-expression (that’s the program) must have, to have a determined value (that’s the output).

In these four papers several different dialects of LISP are studied, and appropriate versions of the halting probability  $\Omega_{\text{LISP}}$  are invented for each of them, together with the corresponding incompleteness theorems. The techniques developed in my complexity theories (A) and (B) are put to good use here.

These LISP  $\Omega$  numbers may not be as random as the fully random  $\Omega$  number in theory (D), but, like the so-called random infinite sequences in my original 1965 randomness paper, they come close. For example, they are Borel normal for blocks of all sizes in any base.

These 1992 papers are immediately included in my second World Scientific volume, *Information-Theoretic Incompleteness* (1992).

I lecture at a meeting on reductionism at Cambridge University. The transcript of that lecture, “Randomness in arithmetic and the decline and fall of reductionism in pure mathematics,” appears later in Cornwell, *Nature’s Imagination*, Oxford University Press, 1995.

- **1995:** I discover how to convert theory (D) into a theory about the size of real programs, programs that you can actually run on a computer

(universal Turing machine) that I simulate using a special version of LISP.

- **Complexity Theory (E)**: Counting the number of bits in a self-delimiting binary program, a bit string with the property that you can tell where it ends by reading it bit by bit without reading a blank endmarker. The program starts with a self-delimiting prefix, indicating which computing machine to simulate, followed by the self-delimiting binary program for that machine, as in theory (D). But in theory (E), (the prefix indicating the machine to simulate) is a LISP S-expression, a program written in a high-level functional programming language, that's converted to a bit string. (E) isn't a new theory, it's a special case of (D) selected because the prefix indicating the machine to simulate is encoded in a particularly convenient manner.

Now, 30 years after starting to work on program-size complexity, I can finally run programs and measure their size.

Several years are needed to complete this concrete version of AIT and to present it in my three Springer-Verlag volumes, *The Limits of Mathematics* (1998), *The Unknowable* (1999), and *Exploring Randomness* (2001). These books come with LISP software and a LISP interpreter.

Honorary doctorate, University of Maine.

- **2000**: Since this year, visiting professor, Computer Science Department, University of Auckland, New Zealand.
- **2002**: Honorary professor, University of Buenos Aires.

Springer-Verlag publishes *Conversations with a Mathematician*, a collection of some of my lecture transcripts and interviews.

I'm invited to present a paper at a philosophy congress in Bonn, Germany, in September. For this purpose, I begin to study philosophy, particularly Leibniz's work on complexity, which I am led to by a hint in a book by Hermann Weyl.

My paper appears two years later (2004) in a proceedings volume published by the Academy Press of the Berlin Academy that was founded by Leibniz. It is reprinted as the second appendix in my book *Meta Math!* (2005).

- **2003:** Lecture notes *From Philosophy to Program Size* published in Estonia, based on a course I give there, winter 2003.
- **2004:** Corresponding member, *Académie Internationale de Philosophie des Sciences*.

Write *Meta Math!*, a high-level popularization of AIT, published the following year by Pantheon Books (2005). This book is not just an explanation of previous work; it presents a *système du monde*.

*Meta Math!* follows Émile Borel in questioning the existence of the bulk of the real numbers, a train of thought further developed in my paper “How real are real numbers?” (*International Journal of Bifurcation and Chaos*, 2006).<sup>22</sup>

- **2005:** I summarize the *système du monde* of *Meta Math!* in the paper “Epistemology as information theory: From Leibniz to  $\Omega$ ” (***Collapse: Journal of Philosophical Research and Development***, 2006).

Honorary president of the scientific committee of the Valparaíso Complex Systems Institute in Chile. Member of the scientific advisory panel of FQXi, devoted to Foundational Questions in Physics & Cosmology.

- **2006:** The centenary of Gödel’s birth. I publish my third *Scientific American* article, on “The limits of reason,” celebrating Leibniz, whom Gödel also admired. This article is translated and published in about a dozen other languages. I summarize my thoughts on incompleteness in an Enriques lecture at the University of Milan, “The halting probability  $\Omega$ : Irreducible complexity in pure mathematics” (*Milan Journal of Mathematics*, 2007).

A collection of some of my philosophy papers is published in Turin. Full member, *Académie Internationale de Philosophie des Sciences*.

- **2007:** World Scientific publishes a more complete collection of my philosophy papers. I establish a connection between the bits of  $\Omega$  and the word problem for semigroups in my paper “An algebraic characterization of the the halting probability” (*Fundamenta Informaticae*, 2007). 60th birthday.

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<sup>22</sup>Vladimir Tasić pointed out to me that Borel has a know-it-all real number—a 1927 version of the  $\Omega$  number. My paper on the ontological status of real numbers is dedicated to Borel’s memory.

To paraphrase Einstein, this timeline will have fulfilled its purpose if it shows how the efforts of a lifetime hang together, and why they lead to certain definite expectations.<sup>23</sup> In particular, I think it would be fruitful to explore the following topics.

## Challenges for the Future

- On the technical side, many questions remain regarding the program-size complexity and the algorithmic probability of computing infinite sets of objects.

More difficult challenges:

- To develop a model of mathematics that is biological, that is, that evolves and develops, that's dynamic, not static. Perhaps a time-dependent formal axiomatic system?
- To understand creativity in mathematics—where do new ideas come from?—and also in biology—how do new, much more complicated organisms develop? Perhaps a life-as-evolving-software model has some merit?
- $\Omega$  = concentrated creativity? Each bit of  $\Omega$  = one bit of creativity? Can human intellectual progress be measured by the number of bits of  $\Omega$  that we know, or are currently capable of knowing, as a function of time?
- Is the universe discrete or continuous? Can physical systems contain an infinite or only a finite number of bits?
- Make a model world in which you can prove life must develop with high probability. It doesn't matter if this world isn't exactly like ours; how can that be important?
- Is the world built out of information, not matter? Is it built out of thought? Is matter just an epiphenomenon, that is, secondary, not primary? And what are thoughts?

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<sup>23</sup>See the final sentence of Einstein's *Autobiographical Notes*.

# Selected Publications by Chaitin

## Non-Technical Books

- *Meta Math!*, Pantheon Books, New York, 2005 (hardcover); Vintage, New York, 2006 (paperback).
- U.K. edition: *Meta Maths*, Atlantic Books, London, 2006.<sup>24</sup>
- *Conversations with a Mathematician*, Springer-Verlag, London, 2002.<sup>25</sup>

## Technical Books

### Lecture Notes

- *From Philosophy to Program Size*, Institute of Cybernetics, Tallinn, 2003.

### Monographs

- *Algorithmic Information Theory*, Cambridge University Press, 1987 (hardcover), 2004 (paperback).
- *The Limits of Mathematics*, Springer-Verlag, Singapore, 1998; reprinted by Springer-Verlag, London, 2002.<sup>26</sup>
- *The Unknowable*, Springer-Verlag, Singapore, 1999.<sup>27</sup>
- *Exploring Randomness*, Springer-Verlag, London, 2001.

### Collections of Papers

- *Information, Randomness and Incompleteness*, World Scientific, Singapore, 1987, 2nd ed., Singapore, 1990.
- *Information-Theoretic Incompleteness*, World Scientific, Singapore, 1992.

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<sup>24</sup>Also published in Greek.

<sup>25</sup>Also published in Japanese and Portuguese.

<sup>26</sup>Also published in Japanese.

<sup>27</sup>Also published in Japanese.

- *Teoria algoritmica della complessità*, Giappichelli Editore, Turin, 2006.
- *Thinking about Gödel and Turing*, World Scientific, Singapore, 2007.