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Name: ______________________________

1. (1 pt.)
   - **Read all material carefully.**
   - *If in doubt whether something is allowed, ask, don't assume.*
   - You may refer to your books, papers, and notes during this test.
   - E-books may be used subject to the restrictions noted in class.
   - No computer or network access of any kind is allowed (or needed).
   - Write, and draw, carefully. Ambiguous or cryptic answers receive zero credit.
   - Use class and textbook conventions for notation, algorithmic options, etc.
   - There is an extra-credit question (marked with $\star$). It is harder than the rest.

   Write your name and group ID (e.g., C3) in the space provided above. The group is for reference only; all work on this quiz is individual work.

2. (15 pts.) There are two distinct (more precisely, nonisomorphic) red-black trees with keys 1 and 2: One with 1 as the root's label and the other with 2 as the root's label. In either case, the root is black and its only nonempty child is red.

   For each value of $n \in \{0, 1, 2, 3, 4, 5\}$, depict all distinct red-black trees with the n keys: $1, 2, \ldots, n$. (The previous paragraph suggests two trees for the case $n = 2$.)

   Explain clearly why the trees you list are the only ones for each value of n.

[additional space for answering the earlier question]

3. (7 pts.) Recall the *triple*-based representation of binary trees from a recent homework. We may use it to represent red-black trees by using the convention of underlining the labels of red nodes.

   Represent each of the trees of Question 2 for the case $n = 5$ in this form. Ensure that the correspondence between the representations in the two questions is clear by numbering your trees consistently.

4. (7 pts.) Suppose we augment the definitions of functions f_7 and f_8 from a recent homework (repeated below for reference) to preserve underlines. That is, the functions leave the underlined or not-underlined states of labels unchanged. Depict, *in both graphical and triple forms*, the result of applying f_7 and f_8, separately, to each of the trees of Question 3.

$$
\begin{aligned}
f_7(t) &= \begin{cases} (n_2, (n_1, l_1, l_2), (n, r_2, r)) & \text{if } t = (n, (n_1, l_1, (n_2, l_2, r_2)), r) \\ \emptyset & \text{otherwise} \end{cases} \\
f_8(t) &= \begin{cases} (n_2, (n, l, l_2), (n_1, r_2, r_1)) & \text{if } t = (n, l, (n_1, (n_2, l_2, r_2), r_1)) \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}
$$

[additional space for answering the earlier question]

5. (10 pts.) Depict the AA tree resulting from the sequential insertion, in the listed order, of

$$8,\ 40,\ 83,\ 77,\ 72,\ 52,\ 62,\ 2,\ 27,\ 25$$

into an empty tree. Depict the state of the tree before and after each insertion, as well as before and after each skew and split operation. Indicate which operations are used and where by annotating the affected trees.

[additional space for answering the earlier question]

$$8,\ 40,\ 83,\ 77,\ 72,\ 52,\ 62,\ 2,\ 27,\ 25$$

[additional space for answering the earlier question]

$$8,\ 40,\ 83,\ 77,\ 72,\ 52,\ 62,\ 2,\ 27,\ 25$$

6. (10 pts.) Depict the AA tree resulting from the sequential deletion, in the listed order, of

$$52,\ 62$$

from the final tree of Question 5. Depict the state of the tree before and after each deletion, as well as before and after each skew and split operations. Indicate which operations are used and where by annotating the affected trees.

[additional space for answering the earlier question]

7. (10 pts.) Repeat Question 5 for a *top-down red-black tree*.

   As in that question, depict the state of the tree before and after each insertion, as well as before and after each rotation and color change. Indicate which operations are used and where by annotating the affected trees.

   8, 40, 83, 77, 72, 52, 62, 2, 27, 25

[additional space for answering the earlier question]

$$8,\ 40,\ 83,\ 77,\ 72,\ 52,\ 62,\ 2,\ 27,\ 25$$

[additional space for answering the earlier question]

$$8,\ 40,\ 83,\ 77,\ 72,\ 52,\ 62,\ 2,\ 27,\ 25$$

8. ⋆ (10 pts.) What is the *maximum* number of insert operations that may be applied to a *bottom-up red-black tree* to yield a bottom-up red-black tree with *at most* four black nodes? (No deletions are permitted.) Reminder: Use the exact definitions and operations from the textbook.

   Justify your answer by providing the sequence of operations and the state of the red-black tree after each operation in the sequence. Clearly indicate the rotations that are performed following each insertion.

   Explain why any longer sequence of insert operations produces more than four black nodes.

[additional space for answering the earlier question]