CARDINALITIES OF $D$-CLASSES IN $S_n$

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Let $S_n$ be the semigroup of all binary relations on a set of $n$ elements. Let $D$ be a $D$-class of $S_n$ with row rank $s$ and column rank $t$ and whose lattice has $p$ elements. Then the number of elements in an $L$-class of $D$ is given by

$$s \sum_{i=0}^{t} (-1)^i \binom{t}{i} (p-1)^n.$$

We find it convenient to phrase the argument in the terminology of matrix theory. We follow the notation introduced in [3] and [4] and use freely the results proved there; otherwise the notation and terminology is that of Clifford and Preston [6].

Let $S$ be the semigroup $S_n$ of all binary relations on a set of $n$ elements, and let the elements of $S$ be represented as $n \times n$ matrices over the Boolean algebra $\{0, 1\}$ of order 2. Such matrices will be called Boolean relation matrices. With each Boolean relation matrix $A$, there is associated a row (column) space $R(A)$ ($C(A)$) and it is well-known that two elements $A$ and $B$ of $S$ are $L$-equivalent iff $R(A) = R(B)$ ($C(A) = C(B)$) [4, 7]. For $A \in S$ the basis of $R(A)$ ($C(A)$) is called the row (column) basis of $A$ and its cardinality is called the row (column) rank of $A$ and denoted by $\rho_r(A)$ ($\rho_c(A)$). It is also known that every finite row (column) space has a unique "basis" [4]. Let $A \in S$. Then $B \in L_A$ ($R_A$) iff
THEOREM 2. Let $A \in S$. Let $s = \rho_e(A)$, $t = \rho_c(A)$, $h = |H_A|$, and $p = |R(A)| = |C(A)|$. Then

\begin{align*}
(i) \quad |L_A| &= \sum_{i=0}^{s} (-1)^{i} (\frac{t}{i})^i (p - i)^n, \\
(ii) \quad |R_A| &= \sum_{i=0}^{t} (-1)^{i} (\frac{t}{i})^i (p - i)^n, \\
(iii) \quad |D_A| &= (|L_A||R_A|)/h.
\end{align*}

Proof. The proofs of (i), (ii), and (iii) follow immediately from Lemma 1 and all the preceding discussions.

COROLLARY 3. If $s = t$, as is the case if $D_A$ is regular, then

$$|D_A| = (|L_A|)^2/h.$$ 

The above results can be formulated in terms of lattices [1].

We conclude this paper by giving an example for counting the values of $p$ in $\Psi(p, k, n)$. If

$$A = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix},$$

then clearly $\rho_e(A) = 4 = \rho_c(A)$, and $|H_A| = 2$. Let

$$J = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{bmatrix},$$

denote an $4 \times 2^4$ matrix in which column represents all possible column vectors for $C(A)$. Then

$$AJ = \begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}.$$ 

Clearly $AJ$ contains 10 different column vectors and so $p = 10$.

Let $J^t$ be the transpose of $J$. Then $J^tA$ contains 10
different row vectors and so \( p = 10 \). Hence we get

\[
|D_A| = (10^n - 4(9)^n + 6(8)^n - 4(7)^n + 6^n)^{2/2}.
\]

REFERENCES


