When buying options on stocks in foreign markets, there are two types to consider. With *quanto* options, an exchange rate between currencies is agreed upon at the beginning by the buyer and seller. This rate is generally(always?) the current exchange rate, and this type of option is preferred when the parties do not wish to be concerned with changes in the exchange rate. The other type of options are called *flexo* options(a.k.a. *composite* options). With flexo options, the payoff is made in the domestic currency, using the exchange rate present at the time of exercise. Thus, the amount of money made by a party exercising the option depends on changes in currency, as well as on the performance of the stock.

Counterintuitively, it is easier to price flexo options than quanto. Let's begin with quanto. We will assume that the stock price S in the foreign currency and the exchange rate X between the foreign and domestic currencies are given by

$$\frac{dS}{S} = (\mu_t^S - q)dt + \sigma_t^S dW_t^S \tag{1}$$

$$\frac{dX}{X} = \mu_t^X dt + \sigma_t^X dW_t^X \tag{2}$$

where q is the dividend yield on S(which lowers the rate of growth of a stock, since the amount of the dividend is deducted from the price of the stock when it is paid out), μ_t^S and μ_t^X are the expected rate of growth of X and S, σ_t^S and σ_t^X are the volatility in S and X, and W_t^S and W_t^S are Brownian motions with correlation coefficient ρ(this means that $E[W_1^S W_1^X] = \rho$. If $W_t^S = W_t^X$ then $\rho = 1$, if $W_t^S = -W_t^X$ then $\rho = -1$, and if W_t^S and W_t^X are independent i.e. unrelated then $\rho = 0$. In real life, this will be a number strictly between -1 and 1, and probably not too far from 0, though in the case of companies who mainly depend on exporting or importing it could be close to -1 or 1, since changes in currency exchange rates can have a large impact on the wellbeing of the business).

If we merely invest in the foreign currency without worrying about S the expected return would be $\mu_t^X + r_f$, where r_f is the interest rate in the foreign foreign country, i.e. r_f is the risk free rate. In order to remove arbitrage opportunities, this must be set equal to the domestic interest rate r_d(why? This makes no sense to me). Thus

$$\mu_t^X = r_d - r_f \tag{3}$$

(So, this seems to say that if there is a difference in the interest rates in the two markets that the currency will generally rise or fall accordingly. Which seems like it cannot be entirely true, because then anyone can make money by just buying currency in the right direction, and I thought we were supposed to be getting rid of arbitrage. Furthermore, there is a countereffect which will drive the exchange rate in the opposite direction, namely that investing in the stock market in a country with a high interest rate becomes less attractive relative to just getting risk free money, thus driving stock prices down in that country, thus messing up the economy and lowering the value of the currency. Why this isn't taken into account I have no idea).

The value of the foreign stock in domestic currency is clearly SX, and the expected return of this is given by

$$\mu_t^S + \mu_t^X + \rho\sigma_t^S\sigma_t^X. \tag{4}$$

(Not quite sure where this is coming from, but I can see that if the two things are positively correlated you should make a bit more, and negatively should be a bit less, b/c for instance $1.1 * 1.1 = 1.21$ but $.9 * .9 = .81$, so you gain 21% but only lose 19%. Conversely $1.1 * .9 = .99 < 1$. But what is the actual derivation?) Again to avoid arbitrage this must be equal to r_d. (Why? Because this isn't risk free, that's just the expected return. I thought arbitrage meant making money without risk) Combining this fact with (3) gives

$$r_S = r_f - \rho\sigma_t^S\sigma_t^X \tag{5}$$

Thus, rewriting (1) using (5) we arrive at

$$\frac{dS}{S} = (r_d - [q - r_f + r_d + \rho\sigma_t^S\sigma_t^X])dt + \sigma_t^S dW_t^S \tag{6}$$

We can then price the quanto option on this stock in the normal way using the Black-Scholes formula, replacing r_s in (1) with r_d, and replacing q in (1) with $q - r_f + r_d + \rho\sigma_t^S\sigma_t^X$. (But why split it up like that? Could just as easily write it as $0 - [q - r_f + \rho\sigma_t^S\sigma_t^X]$ in (5) for instance, why is the way they do it the preferred way?) The result is then multiplied by the exchange rate.

Now for the flexo case. Forgetting about the domestic currency for the moment, the price in the foreign currency is given by the Black-Scholes formula:

$$P_f = Ke^{-r_f T}N(-d_2) - Se^{-qT}N(-d_1) \tag{7}$$

where

- S is the spot price of the stock i.e. price of the stock right now.
- K is the strike price of the stock i.e. the price which the buyer is allowed to buy or sell at when exercising the option.
- r_f is the risk-free interest rate in the foreign market.
- T is the time until expiry.
- q is the dividend yield i.e. the average amount of dividends paid out per unit time. The dividends are assumed to be paid continuously in this model.
- N is the cumulative normal distribution, i.e. $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$. Actually this may be slightly off due to some kind of normalization, the sources I am using are not quite clear.
- $d_1 = \frac{\log \frac{S}{K} + (r_f - q + \sigma_S^2/2)T}{v_S \sqrt{T}}$, where σ_S is the volatility of the underlying(I guess assumed to be a constant here?).
- $d_2 = d_1 - v_s \sqrt{T}$.

This formula is undeniably ugly, but the derivation is not too bad. It uses a lot of arbitrage considerations like above, some standard Brownian motion stuff, and can be found in about a million places on the web. The final price for the flexo option is just the currency exchange rate at time $t = 0$ multiplied by P_f. That is,

$$P_d = X_0 P_f. \tag{8}$$

(Why the expected change in currency isn't taken into account here is a mystery to me. Some anti-arbitrage reason?)