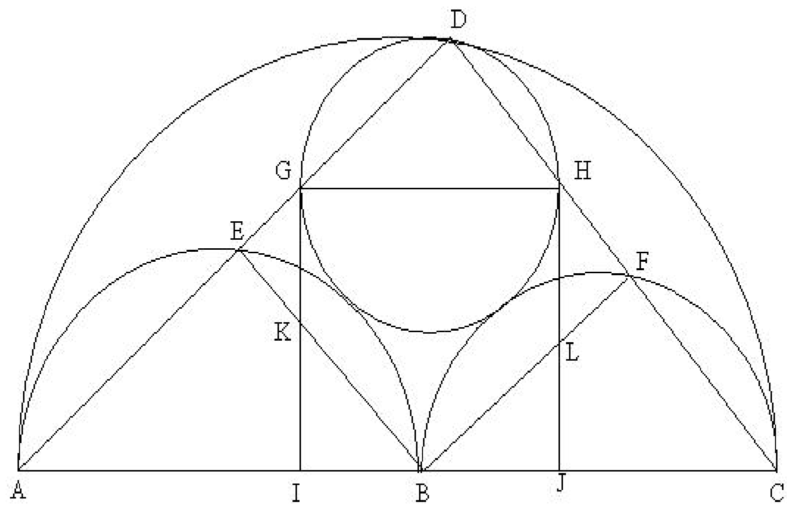
The Shoemaker's Knife: A problem



In the picture above A, B, and C are colinear, and AB, BC, and AC are the diameters of the semicircles shown. AB is not necessarily equal to BC. We inscribe a circle in the curvilinear triangle, and let the point D be the point of tangency between the inscribed circle and the semicircle with diameter AC. Let GH be the diameter of the inscribed circle parallel to AC. Then DHC and DGA are straight lines. Choose I, J on AC so that GI and HJ are perpendicular to AC. Let E, F, K, L be the intersection points, as shown above. Prove the following:

i) $GHIJ$ is a square.

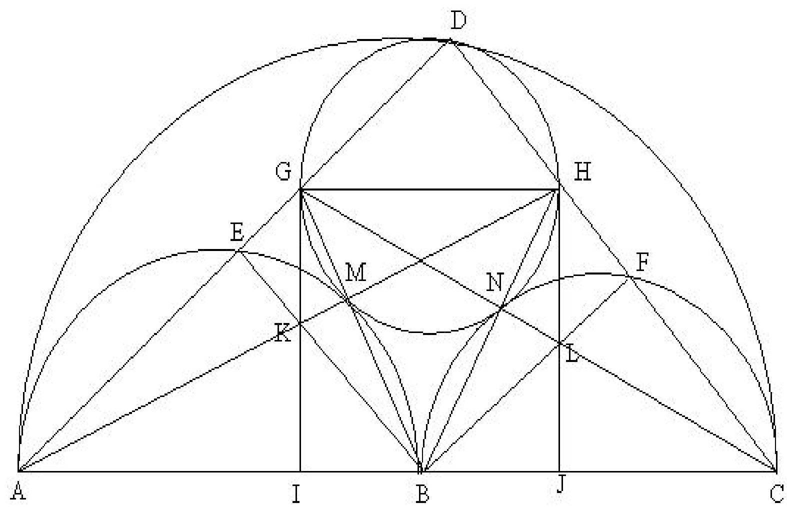
ii) $BEDF$ is a square.

iii) $\frac{AB}{BC} = \frac{IK}{KG} = \frac{BJ}{BI} = \frac{HL}{LJ}$.

Solution: Let $X = AB$ and $Y = BC$. First let us show that

$$GH = IJ = \frac{XY(X+Y)}{XY + X^2 + Y^2} \tag{1}$$

as was shown by Archimedes in his *Book of Lemmas.* Consider the following picture, with M and N the points of tangency of the circles.



By considering successive pairs of similar triangles, we have the following:

$$\frac{AI}{IJ} = \frac{AI}{GH} = \frac{AK}{KH} = \frac{AE}{ED} = \frac{AB}{BC} = \frac{X}{Y} \tag{2}$$

Similarly,

$$\frac{JC}{IJ} = \frac{Y}{X} \tag{3}$$

Thus,

$$X + Y = AI + IJ + JC = IJ(\frac{X}{Y} + 1 + \frac{Y}{X}) \tag{4}$$

This easily simplifies to (1). To get the length of GI, consider the triangle formed by the midpoints of AB, BC, and GH, that is, by the centers of the three circles. This triangle has base $\frac{X+Y}{2}$ and height GI, and therefore has area $\frac{(X+Y)GI}{4}$. Note however that this triangle has side lengths $\frac{X+GH}{2}$, $\frac{Y+GH}{2}$, and $\frac{X+Y}{2}$. By Herron's formula, the area of the triangle is therefore $\frac{1}{4}\sqrt{(X+Y+GH)XY(GH)}$. Plugging the value obtained above for GH and simplifying gives the area as $\frac{XY(X+Y)^2}{4(XY+X^2+Y^2)}$. Equating the two expressions for the area of the triangle gives

$$\frac{1}{4}(X + Y)GI = \frac{XY(X + Y)^2}{4(XY + X^2 + Y^2)} \tag{5}$$

and thus

$$GI = \frac{XY(X + Y)}{4(XY + X^2 + Y^2)} \tag{6}$$

This shows that $GHJI$ is a square, and (i) is proved. Now we'll prove (iii). Note that

$$\frac{KI}{KG} = \frac{AI}{IJ} = \frac{X}{Y} \tag{7}$$

by similar triangles. Likewise for $\frac{HL}{LJ}$. Now,

$$IB = X - AI = X - \frac{X}{Y}\frac{XY(X + Y)}{XY + X^2 + Y^2} = \frac{XY^2}{XY + X^2 + Y^2} \tag{8}$$

Similarly,

$$BJ = \frac{X^2Y}{XY + X^2 + Y^2} \tag{9}$$

Thus, $\frac{IB}{BJ} = \frac{Y}{X}$, and this completes the proof of (iii). To prove (ii), note that triangles AEB and BFC are similar, and similar to BIK and BJL. Thus, $\frac{AE}{EB} = \frac{BF}{FC} = \frac{X}{Y}$. Thus,

$$\frac{BF}{EB} = \frac{AE}{EB}\frac{FC}{EB} = \frac{X}{Y}\frac{Y}{X} = 1 \tag{10}$$

Since $\angle AEB$, $\angle BFC$ and $\angle ADC$ are right angles, we see that $BEDF$ is indeed a square, proving (ii), and we are done. $\square$

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